

On $L(2, 1)$ -labelings of some products of oriented cycles

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Abstract

We refine two results of Jiang, Shao and Vesel on the $L(2, 1)$ -labeling number λ of the Cartesian and the strong product of two oriented cycles. For the Cartesian product, we compute the exact value of $\lambda(\vec{C}_m \square \vec{C}_n)$ for $m, n \geq 40$; in the case of strong product, we either compute the exact value or establish a gap of size one for $\lambda(\vec{C}_m \boxtimes \vec{C}_n)$ for $m, n \geq 48$.

1 Introduction

A $L(p, q)$ -labeling, or $L(p, q)$ -coloring, of a graph G is a function $f : V(G) \rightarrow \{0, \dots, k\}$ such that $|f(u) - f(v)| \geq p$, if $e = uv \in E(G)$; and $|f(u) - f(v)| \geq q$, if there is a path of length two in G joining u and v . To take into account the number of colors used, we say that f is a k - $L(p, q)$ -labeling of G (note that, for historical reasons, the colorings are assumed to start with the label 0). The minimum value of k such that G admits a k - $L(p, q)$ -labeling is denoted by $\lambda_{p,q}(G)$, and it is called the $L(p, q)$ -labeling number of G .

The particular case of $L(p, q)$ -labelings that attracted the most attention is $p = 2$ and $q = 1$, the $L(2, 1)$ -labeling. It was introduced by Yeh [6], and it traces back to the frequency assignment problem of wireless networks introduced by Hale [3]. In this case, we write $\lambda(G)$ instead of $\lambda_{2,1}(G)$ for short.

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The definitions above can be extended to oriented graphs (a directed graph whose underlying graph is simple), namely: if G is an oriented graph, a $L(p, q)$ -labeling of G is a function $f : V(G) \rightarrow \{0, \dots, k\}$ such that $|f(u) - f(v)| \geq p$, if $uv \in E(G)$; and $|f(u) - f(v)| \geq q$, if there is a *directed* path of length two in G joining u and v . The corresponding $L(p, q)$ -labeling number is again denoted by $\lambda_{p,q}(G)$ (in some papers, the notation $\vec{\lambda}_{p,q}(G)$ is used instead). The $L(2, 1)$ -labelings of oriented graphs were first studied by Chang and Liaw [2], and the $L(p, q)$ -labeling problem has been extensively studied since then in both undirected and directed versions. We refer the interested reader to the excellent surveys of Calamoneri [1] and Yeh [7].

In this paper, we study the $L(2, 1)$ -labeling number of the Cartesian and the strong product of two oriented cycles, improving results of Jiang, Shao and Vesel [4]. We use the notation \vec{C}_n to represent the oriented cycle on n vertices, i.e., the digraph such that $V(\vec{C}_n) = \{1, 2, \dots, n\}$ and $E(\vec{C}_n) = \{(1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}$, $n \geq 3$. In the case of Cartesian product, we compute the exact value of $\lambda(\vec{C}_m \square \vec{C}_n)$ for $m, n \geq 40$; in the case of strong product, we either compute the exact value or establish a gap of size one for $\lambda(\vec{C}_m \boxtimes \vec{C}_n)$ for $m, n \geq 48$.

2 Cartesian product

The *Cartesian product* of two graphs (resp. digraphs) G and H is the graph (resp. digraph) $G \square H$ such that $V(G \square H) = V(G) \times V(H)$, and where there is an edge joining (a, x) and (b, y) if $ab \in E(G)$ and $x = y$, or if $a = b$ and $xy \in E(H)$ (resp. there is an edge pointing from (a, x) to (b, y) if $ab \in E(G)$ and $x = y$, or if $a = b$ and $xy \in E(H)$).

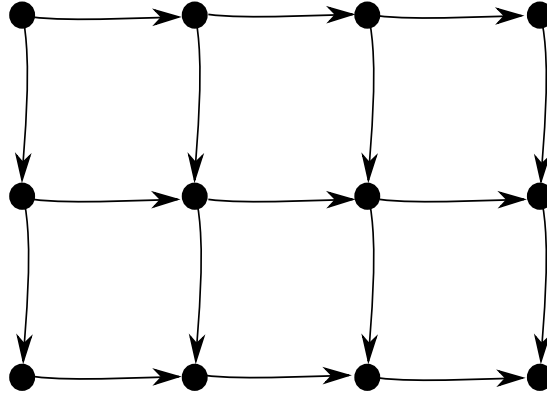


Figure 1: The Cartesian product of \vec{P}_3 and \vec{P}_4

Let $S(m, n) = \{am + bn : a, b \geq 0 \text{ integers not both zero}\}$. A classical result of Sylvester [5] states that $t \in S(m, n)$ for all integers $t \geq (m-1)(n-1)$ that are divisible by $\gcd(m, n)$, the greatest common divisor of m and n .

In [4], Jiang, Shao and Vesel prove the following theorem:

Theorem 1 ([4]). *For all $m, n \in S(5, 11)$, $4 \leq \lambda(\vec{C}_m \square \vec{C}_n) \leq 5$. In particular, the result holds for every $m, n \geq 40$.*

Our result in this section determines the exact value of λ in the range above. We start with a lemma which is a slightly stronger version of Lemma 5 from [4] that can be obtained with the same proof, which we include here for the sake of completeness.

Lemma 1. *For every $m, n \geq 3$ and every 4- $L(2, 1)$ -labeling f of $\vec{C}_m \square \vec{C}_n$, the following periodicity condition holds:*

$$f(i, j) = f(i + 1 \bmod m, j - 1 \bmod n) \text{ for all } i \in [m], j \in [n]. \quad (1)$$

Proof. Let f be a 4- $L(2, 1)$ -labeling of $G = \vec{C}_m \square \vec{C}_n$. We write $f(i, j)$ for the value of f at the vertex (i, j) . By the symmetry of the graph, it is enough to prove that $f(2, 2) = f(1, 3)$. Suppose, for the sake of contradiction, that this is not the case.

Consider the $\vec{P}_2 \square \vec{P}_2$ subgraph spanned by $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$. Every vertex is at distance at most 2 from each other, except for the pair $(2, 2), (1, 3)$. This implies, together with our assumption that $f(2, 2) \neq f(1, 3)$, that every vertex of this subgraph gets a distinct color. It is clear that the color 2 cannot be used in any vertex v of this subgraph, since otherwise the two neighbours of v must receive colors 0 and 4, and there is no color left for the fourth vertex. Thus, the colors used on the vertices of this subgraph are exactly 0, 1, 3 and 4, in some order.

Using the fact that $4 - f$ is a 4- $L(2, 1)$ -labeling of a graph whenever f is, we may assume without loss of generality that $f(2, 2) \in \{0, 1\}$. This implies that $f(1, 3) \in \{0, 1\}$ and $\{f(1, 2), f(2, 3)\} = \{3, 4\}$. If $f(2, 3) = 3$, there is no color for $(3, 3)$, since its within distance two from $(2, 2)$ and $(1, 3)$, colored with 0 and 1, and it is neighbor of a vertex of color 3. If $f(1, 2) = 3$, the same argument applies for the vertex $(1, 1)$. \square

We call labelings with the property of Lemma 1 *diagonal*.

The following lemma from [4] combined with the result of Sylvester will also help us:

Lemma 2. *(Lemmas 2 and 3 in [4]) Let $m, n, p \geq 3$ and $t, k \geq 1$ be integers. If $\lambda(\vec{C}_m \square \vec{C}_n) \leq k$ and $\lambda(\vec{C}_p \square \vec{C}_n) \leq k$, then $\lambda(\vec{C}_{m+tp} \square \vec{C}_n) \leq k$.*

In particular, if m and n are such that $\lambda(\vec{C}_m \square \vec{C}_n) \leq k$, $\lambda(\vec{C}_m \square \vec{C}_m) \leq k$ and $\lambda(\vec{C}_n \square \vec{C}_n) \leq k$, then $\lambda(\vec{C}_a \square \vec{C}_b) \leq k$ for all $a, b \in S(m, n)$, and hence for all $a, b \geq (m-1)(n-1)$ divisible by $\gcd(m, n)$.

Theorem 2. *Let $m, n \geq 40$. Then:*

$$\lambda(\overrightarrow{C_m} \square \overrightarrow{C_n}) = \begin{cases} 4, & \text{if } \gcd(m, n) \geq 3; \\ 5, & \text{otherwise.} \end{cases}$$

Proof. For $m, n \geq 3$, let G denote the graph $\overrightarrow{C_m} \square \overrightarrow{C_n}$, i.e., $V(G) = [m] \times [n]$ and the directed edges of G point from (i, j) to $(i + 1 \bmod m, j)$ and to $(i, j + 1 \bmod n)$, for every $i \in [m], j \in [n]$. For a labeling f , we write $f(i, j)$ instead of $f((i, j))$ for short.

Let $d = \gcd(m, n)$ and assume first that $d \notin \{1, 2\}$. According to Lemma 2, it is enough to prove that $\lambda(\overrightarrow{C_d} \square \overrightarrow{C_d}) = 4$. Any $4-L(2, 1)$ -labeling f of $\overrightarrow{C_d}$ can be extended to a $4-L(2, 1)$ -labeling f' of $\lambda(\overrightarrow{C_d} \square \overrightarrow{C_d})$ by setting $f'(i, j) = f(i + j \bmod d)$. It suffices to show, then, that $\lambda(\overrightarrow{C_d}) = 4$.

If $d \equiv 0 \pmod{3}$, then we can label $\overrightarrow{C_d}$ with $d/3$ blocks 024. If $d \equiv 1 \pmod{3}$, we label $\overrightarrow{C_d}$ with $(d - 4)/3$ consecutive blocks 024 and then one block 0314. Finally, if $d \equiv 2 \pmod{3}$, then we label $\overrightarrow{C_d}$ with $(d - 2)/3$ consecutive blocks 024 and then a block 13.

On the other hand, assume for the sake of contradiction that $d \in \{1, 2\}$ and there is a $4-L(2, 1)$ -labeling f of $\overrightarrow{C_m} \square \overrightarrow{C_n}$. In particular, $m \neq n$, so let us assume that $m > n$.

It is easy to check that, if $m \geq n + 3$, f induces a valid $4-L(2, 1)$ -labeling of $\overrightarrow{C_{m-n}} \square \overrightarrow{C_n}$. In fact, let $g(i, j) = f(i, j)$ for all $1 \leq i \leq m - n$ and $1 \leq j \leq n$. We claim that g is a $4-L(2, 1)$ -labeling of $\overrightarrow{C_{m-n}} \square \overrightarrow{C_n}$, which, in particular, satisfies (1) as well.

Indeed, all we have to check is that the following conditions hold for g , since the other restrictions are inherited by f : $|g(m - n - 1, j) - g(1, j)| \geq 1$, $|g(m - n, j) - g(1, j)| \geq 2$, $|g(m - n, j) - g(2, j)| \geq 1$, $|g(m - n, j) - g(1, j + 1 \bmod n)| \geq 2$, for every $j \in [n]$. All these conditions follow from $g(m - n - 1, j) = f(m - n - 1, j) = f(m - 1, j + n \bmod n) = f(m - 1, j)$ and $g(m - n, j) = f(m - n, j) = f(m, j + n \bmod n) = f(m, j)$, which result from the application of (1) n times, together with the fact that f is a $L(2, 1)$ -labeling of $\overrightarrow{C_m} \square \overrightarrow{C_n}$.

Applying this argument consecutively, using the fact that $d = \gcd(m, n)$ and by the symmetry of the factors of the product, we conclude that f induces a $4-L(2, 1)$ -labeling c of either $\overrightarrow{C_{k+1}} \square \overrightarrow{C_k}$ or $\overrightarrow{C_{k+2}} \square \overrightarrow{C_k}$, for some $k \geq 3$. This is a contradiction, since in this case we would have, by Lemma 1, $c(1, 1) = c(2, k) = \dots = c(k + 1, 1)$ and $(k + 1, 1)$ and $(1, 1)$ are joined by an edge or by a directed path of length two, respectively.

□

3 Strong product

The *strong product* of two graphs (resp. digraphs) G and H is the graph (resp. digraph) $G \boxtimes H$ such that $V(G \boxtimes H) = V(G) \times V(H)$, and where there is an edge joining (a, x) and (b, y) if either $ab \in E(G)$ and $x = y$, or if $a = b$ and $xy \in E(H)$, or if $ab \in E(G)$ and $xy \in E(H)$. (resp. either there is an edge pointing from (a, x) to (b, y) if $ab \in E(G)$ and $x = y$, or if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $xy \in E(H)$).

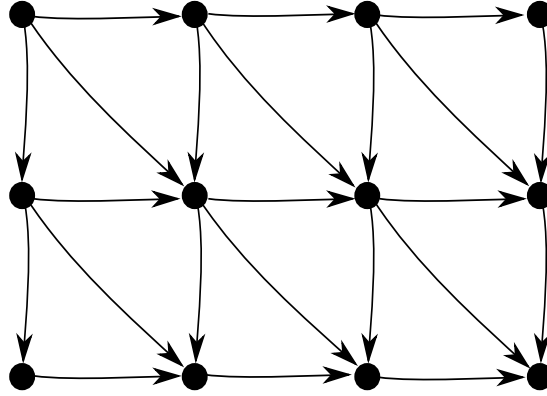


Figure 2: The strong product of \vec{P}_3 and \vec{P}_4

In the same paper, Jiang, Shao and Vesel prove the following result for the strong product of two directed cycles:

Theorem 3 ([4]). *If $m, n \geq 48$, then $6 \leq \lambda(\vec{C}_m \boxtimes \vec{C}_n) \leq 8$.*

In this section, we refine this theorem in the following way:

Theorem 4. *If $m, n \geq 48$, then*

$$\lambda(\vec{C}_m \boxtimes \vec{C}_n) = \begin{cases} 6, & \text{if } m \equiv n \equiv 0 \pmod{7}; \\ 7 \text{ or } 8, & \text{otherwise.} \end{cases}$$

The key lemma of in the proof of Theorem 4 is analogous to Lemma 1:

Lemma 3. *Let $m, n \geq 4$ be integers. Any 6-L(2,1)-labeling f of $\vec{C}_m \boxtimes \vec{C}_n$ is diagonal, i.e., the following condition holds:*

$$f(i, j) = f(i + 1 \bmod m, j - 1 \bmod n) \text{ for all } i \in [m], j \in [n]. \quad (2)$$

Proof of Lemma 3. Let G be the graph $\vec{C}_m \boxtimes \vec{C}_n$. For every vertex (i, j) of G , there is a $\vec{P}_4 \boxtimes \vec{P}_4$ subgraph of G as in Figure 3 such that v_{23} is the vertex (i, j) .

It suffices, then, to show that $f(v_{23}) = f(v_{32})$ for every 6- $L(2, 1)$ -labeling of the graph in Figure 3.

Let f be a 6- $L(2, 1)$ -labeling of G . By the fact that $6 - f$ is also a 6- $L(2, 1)$ -labeling of G , we may assume that $f(v_{32}) \in \{0, 1, 2, 3\}$. We will divide the rest of the proof in cases according to the value of $f(v_{32})$. In each case, we will assume that $f(v_{23}) \neq f(v_{32})$ and reach a contradiction by applying the rules of $L(2, 1)$ -labeling and finding a vertex for which there is no available color. We will use the notation $v!$ to mean that there is no color available for the vertex v , and the notation $\{u, v\} \in S!$ to mean that u and v cannot be colored using the colors in S , where S is the set of possible colors for u and v based on the colors of the previous vertices and the rules of $L(2, 1)$ -labeling. For instance, if u and v are joined by an edge, then $\{u, v\} \in \{0, 1\}!$.

Case 1: $f(v_{32}) = 3$

- $f(v_{23}) \in \{5, 6\} \Rightarrow \{v_{22}, v_{33}\} \in \{0, 1\}!$.
- $f(v_{23}) = 4 \Rightarrow f(v_{22}) = 6$ and $f(v_{33}) \in \{0, 1\}$, or $f(v_{22}) \in \{0, 1\}$ and $f(v_{33}) = 6$. In the first case, $f(v_{43}) = 5 \Rightarrow f(v_{44}) = 2 \Rightarrow f(v_{33}) = 0 \Rightarrow v_{34}!$; in the second, $f(v_{21}) = 5 \Rightarrow f(v_{11}) = 2 \Rightarrow f(v_{22}) = 0 \Rightarrow v_{12}!$.
- $f(v_{23}) \in \{0, 1\} \Rightarrow \{v_{12}, v_{23}\} \in \{5, 6\}!$.
- $f(v_{23}) = 2 \Rightarrow f(v_{22}) = 0$ and $f(v_{33}) \in \{5, 6\}$, or $f(v_{33}) = 0$ and $f(v_{22}) \in \{5, 6\}$. In the first case, $\Rightarrow f(v_{43}) = 1 \Rightarrow f(v_{44}) = 4 \Rightarrow f(v_{33}) = 6 \Rightarrow v_{34}!$; in the second, $\Rightarrow f(v_{21}) = 1 \Rightarrow f(v_{11}) = 4 \Rightarrow f(v_{22}) = 6 \Rightarrow v_{12}!$.

Case 2: $f(v_{32}) = 0$

- $f(v_{23}) = 3 \Rightarrow \{v_{22}, v_{33}\} \in \{5, 6\}!$.
- $f(v_{23}) = 5 \Rightarrow \{v_{22}, v_{33}\} \in \{2, 3\}!$.
- $f(v_{23}) = 2 \Rightarrow f(v_{22}) = 6$ and $f(v_{33}) = 4$, or $f(v_{33}) = 6$ and $f(v_{22}) = 4$. In the first case, $v_{43}!$; in the second, $v_{21}!$.
- $f(v_{23}) = 4 \Rightarrow f(v_{22}) = 6$ and $f(v_{33}) = 2$, or $f(v_{22}) = 2$ and $f(v_{33}) = 6$. In the first case, $v_{34}!$; in the second, $v_{12}!$.
- $f(v_{23}) = 6 \Rightarrow f(v_{22}) = 2$ and $f(v_{33}) = 4$, or $f(v_{22}) = 4$ and $f(v_{33}) = 2$. In the first case, $v_{43}!$; in the second, $v_{21}!$.
- $f(v_{23}) = 1 \Rightarrow f(v_{22}) = 3$ and $f(v_{33}) = 5$, or $f(v_{22}) = 5$ and $f(v_{33}) = 3$, or $f(v_{22}) = 6$ and $f(v_{33}) = 4$, or $f(v_{22}) = 4$ and $f(v_{33}) = 6$, or $f(v_{22}) = 6$ and $f(v_{33}) = 3$, or $f(v_{22}) = 3$ and $f(v_{33}) = 6$. In the first and the third cases, $v_{34}!$; in the second and the fourth, $v_{12}!$; in the fifth, $f(v_{34}) = 5 \Rightarrow v_{44}!$; in the sixth, $f(v_{12}) = 5 \Rightarrow v_{11}!$.

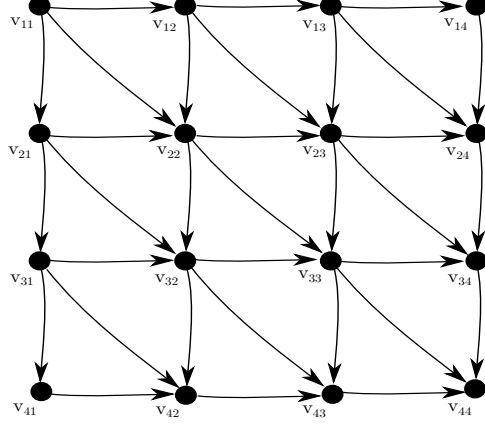


Figure 3: A $\vec{P}_4 \boxtimes \vec{P}_4$ subgraph of $\vec{C}_m \boxtimes \vec{C}_n$

Case 3: $f(v_{32}) = 1$

- $f(v_{23}) = 3$ or $f(v_{23}) = 0$: can be treated similarly as the previous cases $f(v_{32}) = 3, f(v_{23}) = 1$, and $f(v_{32}) = 0, f(v_{23}) = 1$.
- $f(v_{23}) = 6 \Rightarrow \{v_{22}, v_{33}\} \in \{3, 4\}!$.
- $f(v_{23}) = 5 \Rightarrow \{v_{22}, v_{33}\} \in \{3\}!$.
- $f(v_{23}) = 4 \Rightarrow \{v_{22}, v_{33}\} \in \{6\}!$.
- $f(v_{23}) = 2 \Rightarrow f(v_{22}) = 4$ and $f(v_{33}) = 6$, or $f(v_{22}) = 6$ and $f(v_{33}) = 4$. In the first case, $f(v_{43}) = 3$ and $f(v_{34}) = 0 \Rightarrow v_{44}!$; in the second, $f(v_{21}) = 3$ and $f(v_{12}) = 0 \Rightarrow v_{11}!$.

Case 4: $f(v_{32}) = 2$

- $f(v_{23}) \in \{0, 1, 3\}$: can be treated similarly as in previous cases with $f(v_{23})$ and $f(v_{32})$ swapped.
- $f(v_{23}) = 5 \Rightarrow \{v_{22}, v_{33}\} \in \{0\}!$.
- $f(v_{23}) = 6 \Rightarrow f(v_{22}) = 0$ and $f(v_{33}) = 4$, or $f(v_{22}) = 4$ and $f(v_{33}) = 0$. In the first case, $v_{43}!$; in the second, $v_{21}!$.
- $f(v_{23}) = 4 \Rightarrow f(v_{22}) = 0$ and $f(v_{33}) = 6$, or $f(v_{22}) = 6$ and $f(v_{33}) = 0$. In the first case, $v_{43}!$; in the second, $v_{21}!$.

□

Proof of Theorem 4. By Theorem 3, it is enough to prove that $G = \lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 6$ if and only if 7 divides m and n .

If both m and n are divisible by 7, the following periodic labeling is easily checked to be a $L(2, 1)$ -labeling of $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$: the pattern 0246135 is repeated along the cycles. More explicitly, $f(i, j) = 0, 2, 4, 6, 1, 3, 5$ if $i + j \equiv 2, 3, 4, 5, 6, 0, 1 \pmod{7}$, respectively.

On the other hand, assume that G admits a $6-L(2, 1)$ -labeling f . By Lemma 3, f is diagonal. Similarly as in the proof of Theorem 2, it is simple to check that, if $m \geq n + 3$, then f induces a $6-L(2, 1)$ -labeling of $\overrightarrow{C_{m-n}} \boxtimes \overrightarrow{C_n}$, simply by considering the restriction of the coloring in a $\overrightarrow{C_{m-n}} \boxtimes \overrightarrow{C_n}$ subgraph of $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$. Again, applying this argument consecutively, we are either left with a $\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}$, where $d = \gcd(m, n)$, if $d \geq 3$, or a $\overrightarrow{C_{k+1}} \boxtimes \overrightarrow{C_k}$, or a $\overrightarrow{C_{k+2}} \boxtimes \overrightarrow{C_k}$.

In the last two cases, the fact that f is diagonal immediately implies that there are either two consecutive vertices or two vertices within distance two which receive the same color, which is a contradiction.

We are done, then, if we prove that $\lambda(\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}) \geq 7$ if d is not a multiple of 7. Indeed, assume that there is a $6-L(2, 1)$ -labeling of $\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}$. Again, by Lemma 3, it should be diagonal. In particular, it means that it corresponds to a labeling of the cycle C_d with the following property: every pair of vertices with distance at most two must receive colors two apart, and every pair of vertices with distance three or four must receive distinct colors: indeed, the vertex $(i + 1, j)$ is a neighbor of (i, j) , and the vertex $(i + 2, j)$ has the same color as $(i + 1, j + 1)$, which is adjacent to (i, j) in $\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}$, so they must receive colors two apart from the color of (i, j) ; similarly we prove that $(i + 3, j)$ and $(i + 4, j)$ must receive distinct colors from (i, j) . Such a coloring is denoted in the literature by $L(2, 2, 1, 1)$ -labeling. Note that, in particular, this implies the statement for $d = 3$ and $d = 4$. Let us assume in what follows that $d \geq 5$.

If C_d has such a coloring, it is readily checked that it must use the color 0 or 6. Indeed, otherwise all available colors are 12345, and it is impossible to color a P_5 subgraph of C_d with only these colors.

By symmetry, we may assume that 0 is used. Let c be the coloring of C_d , and let the integers modulo d represent its vertices. Let us assume that $c(0) = 0$.

If $c(1) = 3$, then 2 must receive a color from $\{5, 6\}$. If $c(2) = 5$, then $c(3) = 1$, and then $c(-1) = 6$, $c(-2) = 2$ or $c(-2) = 4$. In the first case, $c(-3) = 4$, and finally there is no available color for -4 ; in the second, $c(-3) \in \{1, 2\}$, and in either case there is no color available for -4 . If $c(2) = 6$, then $c(3) = 1$, and then $c(-1) = 5$, $c(-2) = 2$, and there is no available color for -3 .

If $c(1) = 4$, then $c(2)$ is either 2 or 6. In the first case, $c(-1) = 6$ and there is no available color for 3. In the second, $c(-1) = 2$, which implies $c(-2) = 5$ and then there is no available color for -3 .

If $c(1) = 6$, then $c(2) \in \{2, 3, 4\}$. The first implies $c(3) = 4$ and there is no color for 4. The second implies $c(3) = 1$, and then $c(4) = 5$ and there is no color for 5. Finally, the third implies that either $c(3) = 1$ or $c(3) = 2$, both of which makes impossible to find a color for 4.

The argument above implies that the neighbors of 0 must have colors 2 and 5. Without loss of generality, we may assume that $c(1) = 5$ and $c(-1) = 2$. This implies that $c(2) = 3$, which in turn implies $c(3) = 1$, then $c(4) = 6$ and $c(5) = 4$. It follows that $c(6) \in \{0, 2\}$, but by the paragraphs above, 4 cannot be a neighbor of 0, so $c(6) = 2$. Then $c(7) = 0$, and the block 2053164 of size 7 is repeated. The only way the coloring can be completed along the cycle is, then, if 7 divides d . □

4 Final remarks

The natural next step would be to close the gap left from Theorem 4, deciding for which m and n we have $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 7$.

In the proof of Theorem 4, we gave a periodic 6-labeling of $\lambda(\overrightarrow{C_7} \boxtimes \overrightarrow{C_7})$, namely that one in which the pattern 0246135 is repeated along the cycles diagonally. In a similar fashion, the following periodic 7-coloring works for $\lambda(\overrightarrow{C_8} \boxtimes \overrightarrow{C_8})$: 02461357. Concatenating these two patterns, one can show that $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_m}) = 7$ for every sufficiently large m (namely, for every $m \in S(7, 8)$; in particular for $m \geq 42$), and consequently $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 7$ for every m, n such that 7 does not divide both m and n and $\gcd(m, n) \geq 42$.

Finally, we remark that it is simple to check that the proof of Lemma 2 works in the setting of strong product of cycles as well. As we know from the paragraph above that $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_m}) = 7$ for every m in $S(7, 8)$ (and, in particular for every $m \geq 42$), to prove that $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 7$ for all sufficiently large m and n , it would be enough to find a pair of coprime integers $a, b \in S(7, 8)$ such that $\lambda(\overrightarrow{C_a} \boxtimes \overrightarrow{C_b}) = 7$.

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